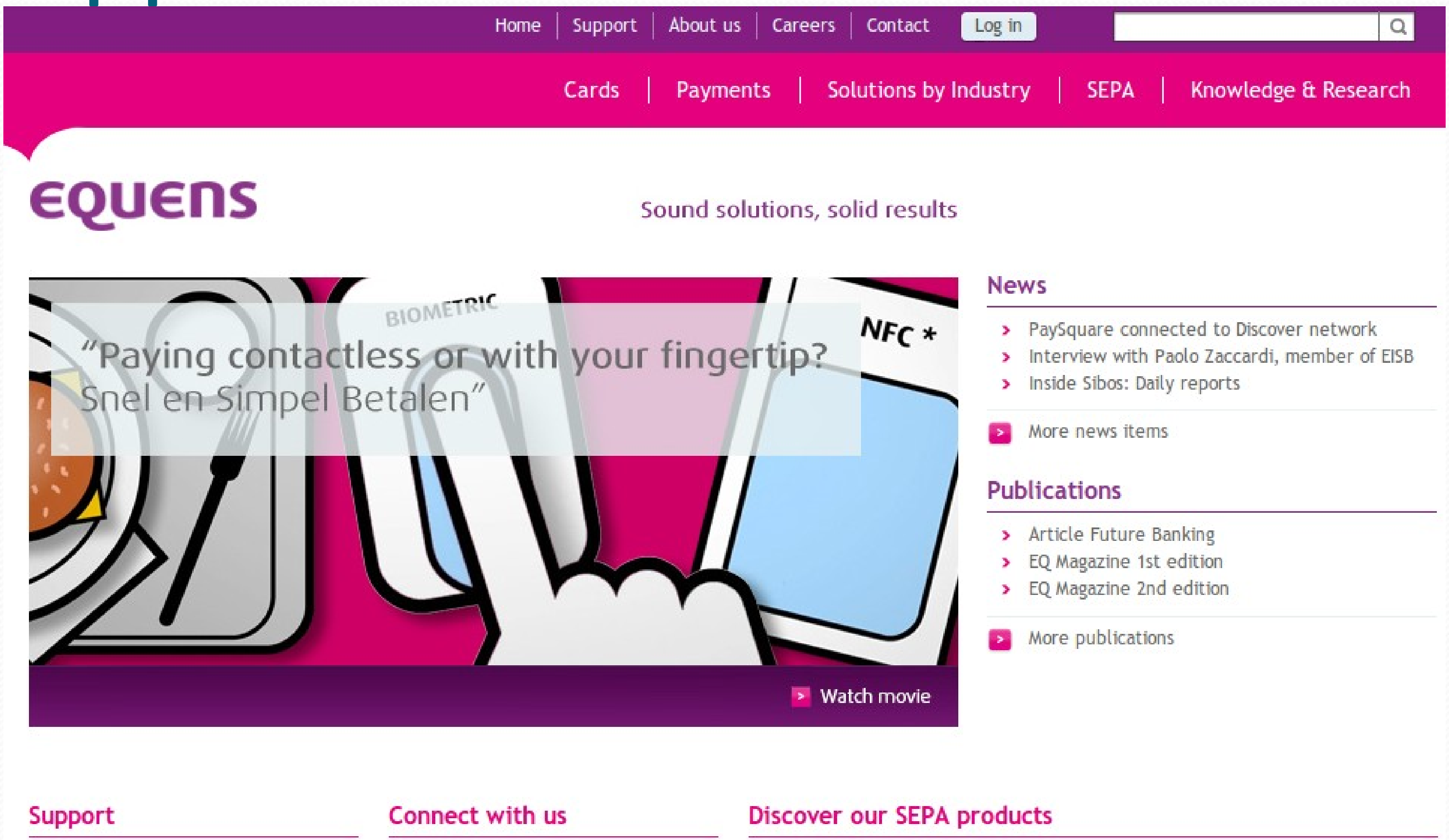
Fuzzy Sets and Fuzzy Logic

Application



Application

(From a press release)

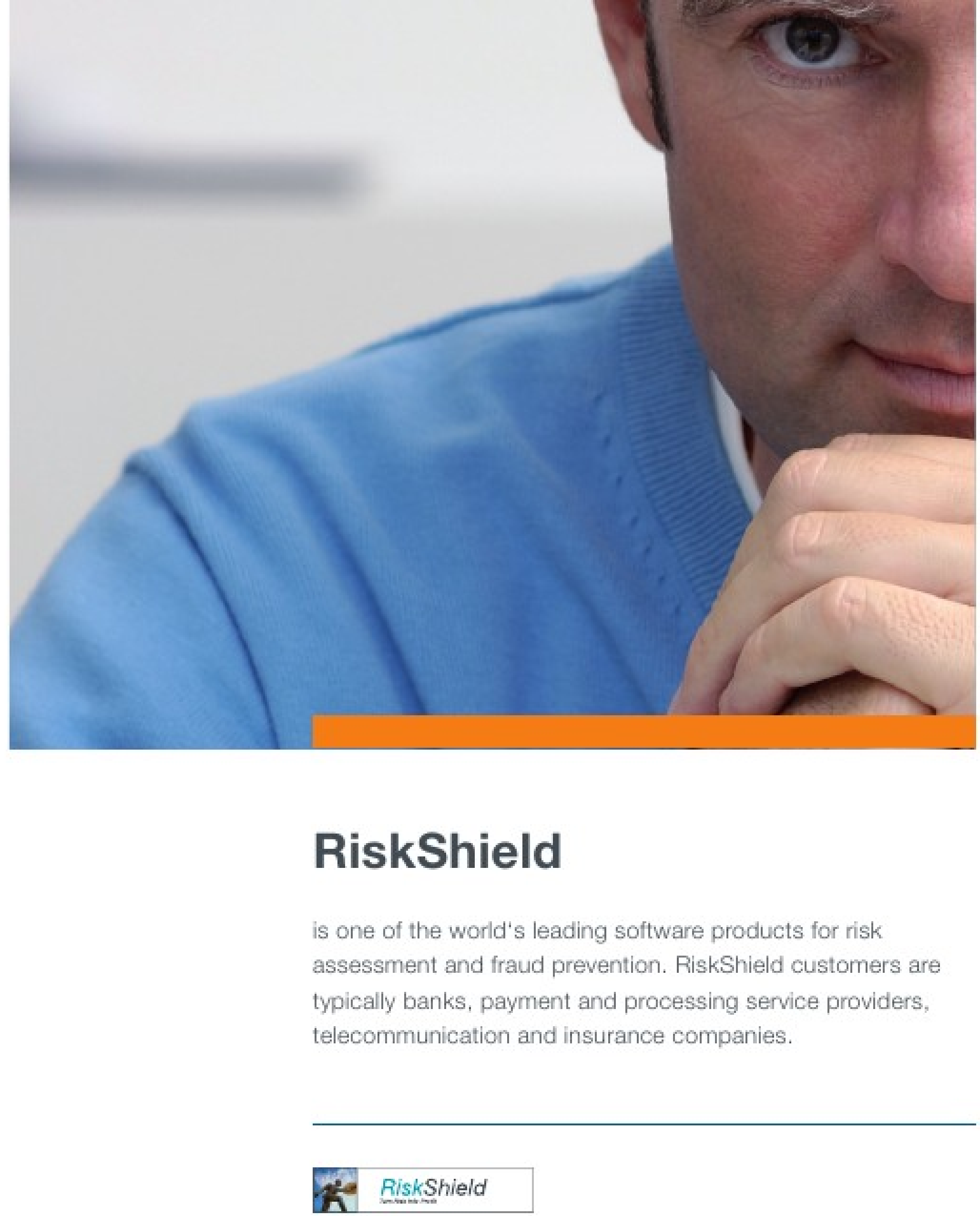
Equens to offer RiskShield Fraud Protection for Card Payments

Today Equens, one of the largest pan-European card and payment Processors, announced that it has selected RiskShield from INFORM GmbH as the basis for a new approach to fraud detection and behaviour monitoring. By utilising the flexibility offered by RiskShield, Equens will be able to offer tailor-made fraud management services to issuers and acquirers.

UTRECHT, The Netherlands, 30/10/2012

Application

- From the brochure of “RiskShield”



Application

ADVANTAGES OF

RiskShield

Based on Fuzzy Logic
Evaluates historical data such as typical user or contractual partner behaviour
Maximum performance (decisions within milliseconds)
High transparency for decisions
Short reaction times to new risk and fraud patterns
Reduces losses due to fraud or a missing risk assessment that can amount to millions

Fuzzy sets on discrete universes

- Fuzzy set C = "desirable city to live in"

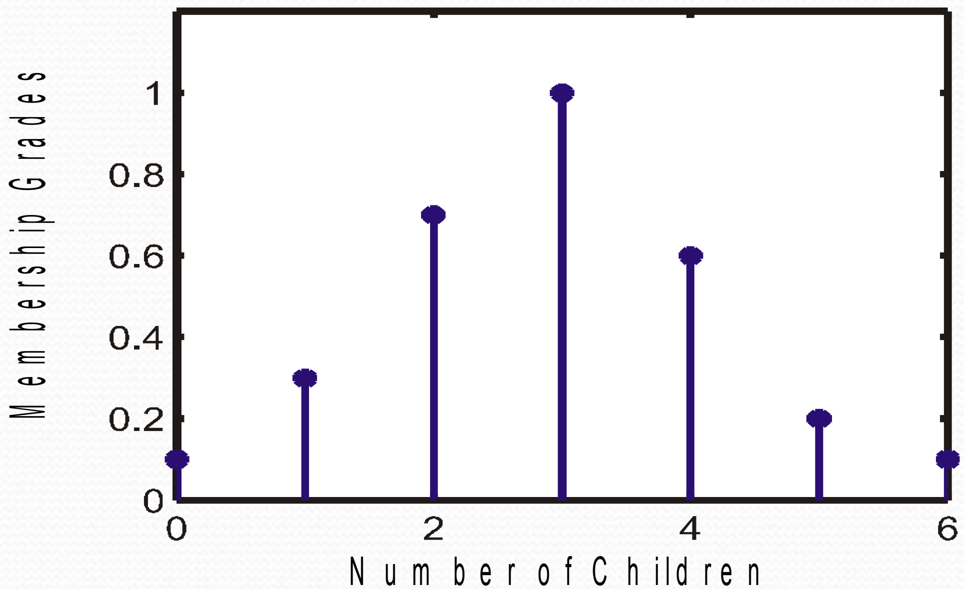
  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

- Fuzzy set A = "sensible number of children"

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



Fuzzy sets & fuzzy logic

- Fuzzy sets can be used to define a level of truth of facts
- Fuzzy set C = "desirable city to live in"

  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

corresponds to a fuzzy interpretation in which
C(SF) is true with degree 0.9
C(Boston) is true with degree 0.8
C(LA) is true with degree 0.6
→ membership function $\mu_C(x)$ can be seen as a (fuzzy) predicate.

Fuzzy logic formulas

- Membership functions: $\mu_C(x), \mu_B(x)$
  - B="City is beautiful"
  - C="City is clean"
- Formulas: $\mu_C(x) \wedge \neg\mu_B(x), \mu_C(x) \vee \mu_B(x), \ldots$
- What is the truth value of such formulas for given *x*?
- We need to define a meaning for the connectives

Fuzzy logic formulas

- Standard interpretations of connectives in fuzzy logic:
  - Negation: $\neg \mu_A(x) = 1 - \mu_A(x)$
  - Conjunction: $\mu_A(x) \wedge \mu_B(y) = \min(\mu_A(x), \mu_B(y))$
  - Disjunction: $\mu_A(x) \vee \mu_B(y) = \max(\mu_A(x), \mu_B(y))$

Generalized negation

- General requirements:
  - Boundary: N(0)=1 and N(1) = 0
  - Monotonicity: N(a) > N(b) if a < b
  - Involution: N(N(a)) = a
- Two types of fuzzy complements:
  - Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

  - Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$

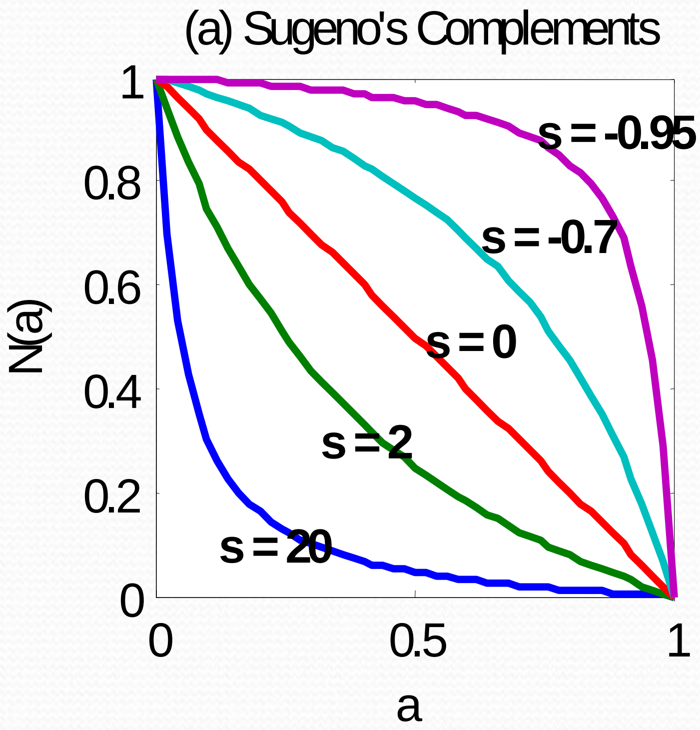
Sugeno's and Yager's complements

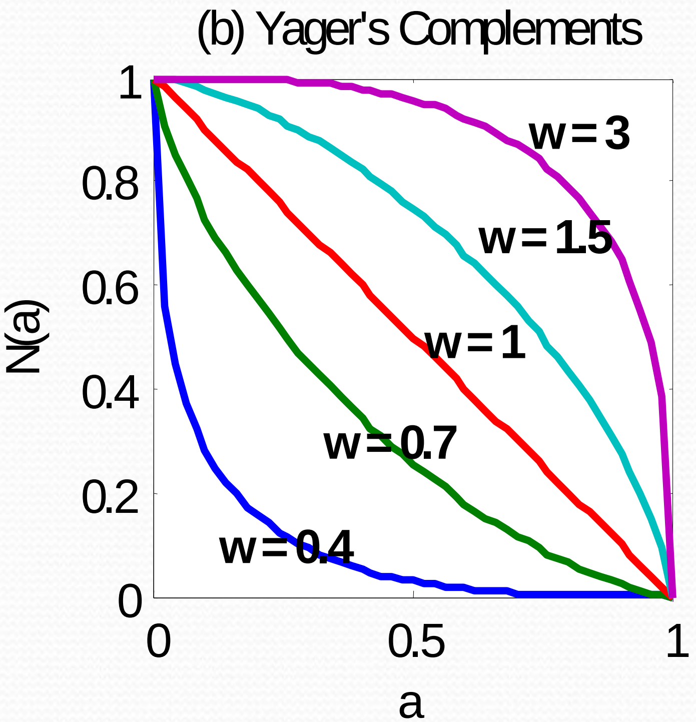
Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$





Generalized intersection (Triangular/T-norm, logical and)

- Basic requirements:
  - Boundary: $T(0, a) = T(a,0) = 0$, $T(a, 1) = T(1, a) = a$
  - Monotonicity: $T(a, b) <= T(c, d)$ if $a <= c$ and $b <= d$
  - Commutativity: $T(a, b) = T(b, a)$
  - Associativity: $T(a, T(b, c)) = T(T(a, b), c)$

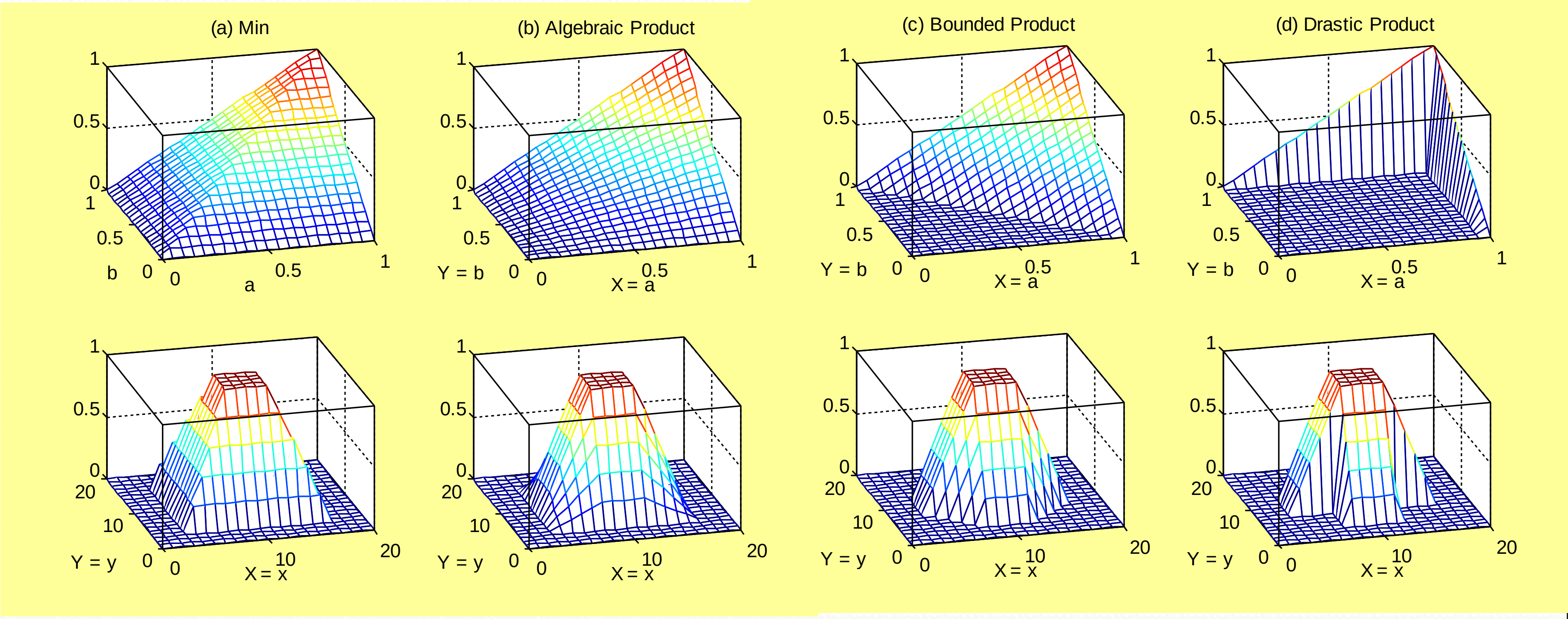
Generalized intersection (Triangular/T-norm)

- Examples:
  - Minimum: $T(a,b) = min(a,b)$
  - Algebraic product: $T(a,b) = a \cdot b$
  - Bounded product: $T(a,b) = max(0,(a+b-1))$
  - Drastic product:

$$T(a,b) = \begin{cases} a & if\ b=1 \\ b & if\ a=1 \\ 0 & otherwise \end{cases}$$

T-norm operator

Minimum: $T_m(a, b)$ $\geq$ **Algebraic product: $T_a(a, b)$** $\geq$ **Bounded product: $T_b(a, b)$** $\geq$ **Drastic product: $T_d(a, b)$**

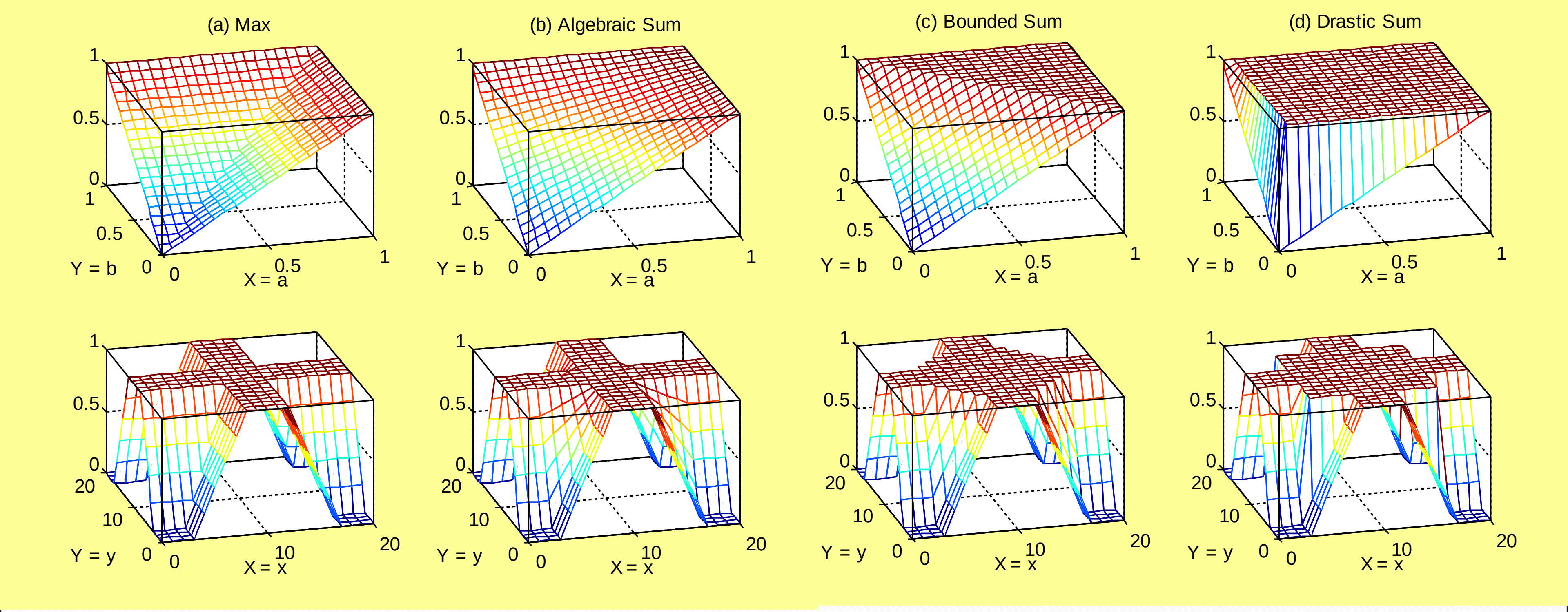


Generalized union (t-conorm)

- Basic requirements:
  - Boundary: S(1, a) = 1, S(a, 0) = S(0, a) = a
  - Monotonicity: S(a, b) < S(c, d) if a < c and b < d
  - Commutativity: S(a, b) = S(b, a)
  - Associativity: S(a, S(b, c)) = S(S(a, b), c)
- Examples:
  - Maximum: $S(a,b)=max(a,b)$
  - Algebraic sum: $S(a,b)=a+b-a\cdot b$
  - Bounded sum: $S(a,b)=min(1,(a+b))$
  - Drastic sum

T-conorm operator

Maximum: $S_m(a, b)$ $\leq$ **Algebraic sum: $S_a(a, b)$** $\leq$ **Bounded sum: $S_b(a, b)$** $\leq$ **Drastic sum: $S_d(a, b)$**



Generalized De Morgan's Law

- T-norms and T-conorms are duals which support the generalization of DeMorgan's law:
  - $T(a, b) = N(S(N(a), N(b)))$
  - $S(a, b) = N(T(N(a), N(b)))$

$$T_m(a, b) \longleftrightarrow S_m(a, b)$$
$$T_a(a, b) \longleftrightarrow S_a(a, b)$$
$$T_b(a, b) \longleftrightarrow S_b(a, b)$$
$$T_d(a, b) \longleftrightarrow S_d(a, b)$$

Fuzzy if-then rules

- General format:

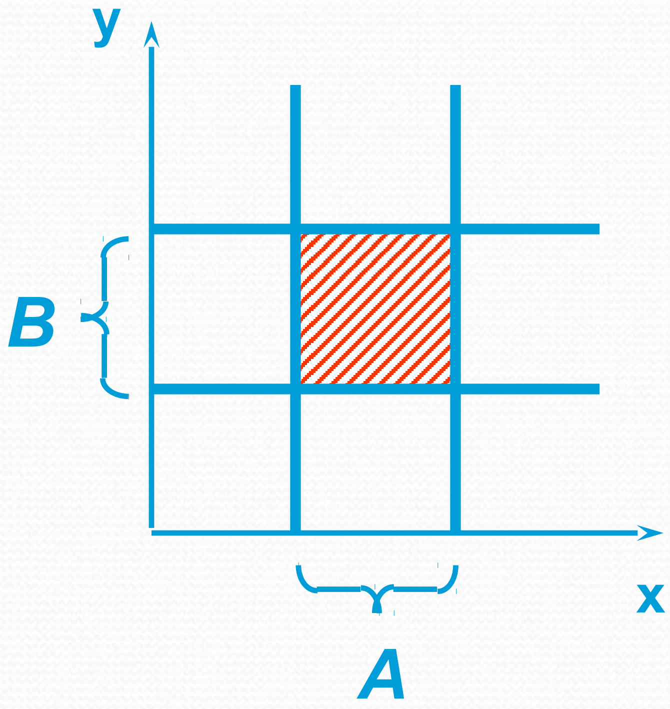
  If x is A then y is B

- Examples:
  - If pressure is high, then volume is small
  - If a restaurant is expensive, then order small dishes
  - If a tomato is red, then it is ripe
  - If the speed is high, then apply the brake a little

Interpretation of Implication

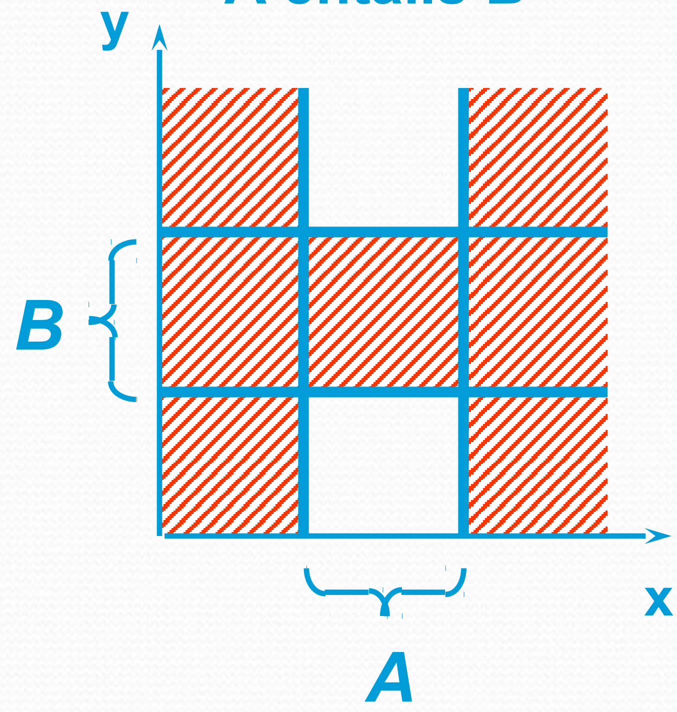
Common in Fuzzy logic

A coupled with B



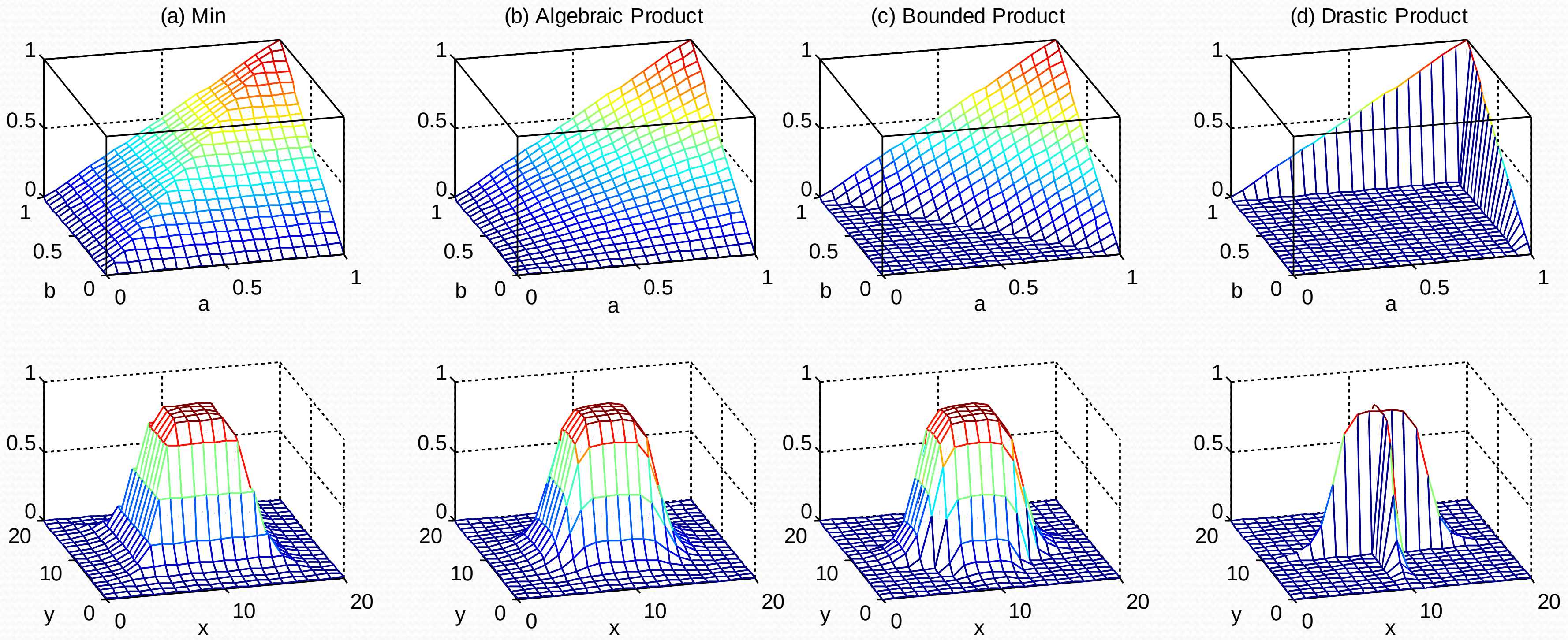
Implication in traditional logic

A entails B



A coupled with B

Use the T-norm...



A entails B

- Boolean fuzzy implication (based on $\neg A \vee B$)

$$m_R(x,y) = max(1 - m_A(x), m_B(y))$$

- Zadeh's max-min implication (based on $\neg A \vee (A \wedge B)$)

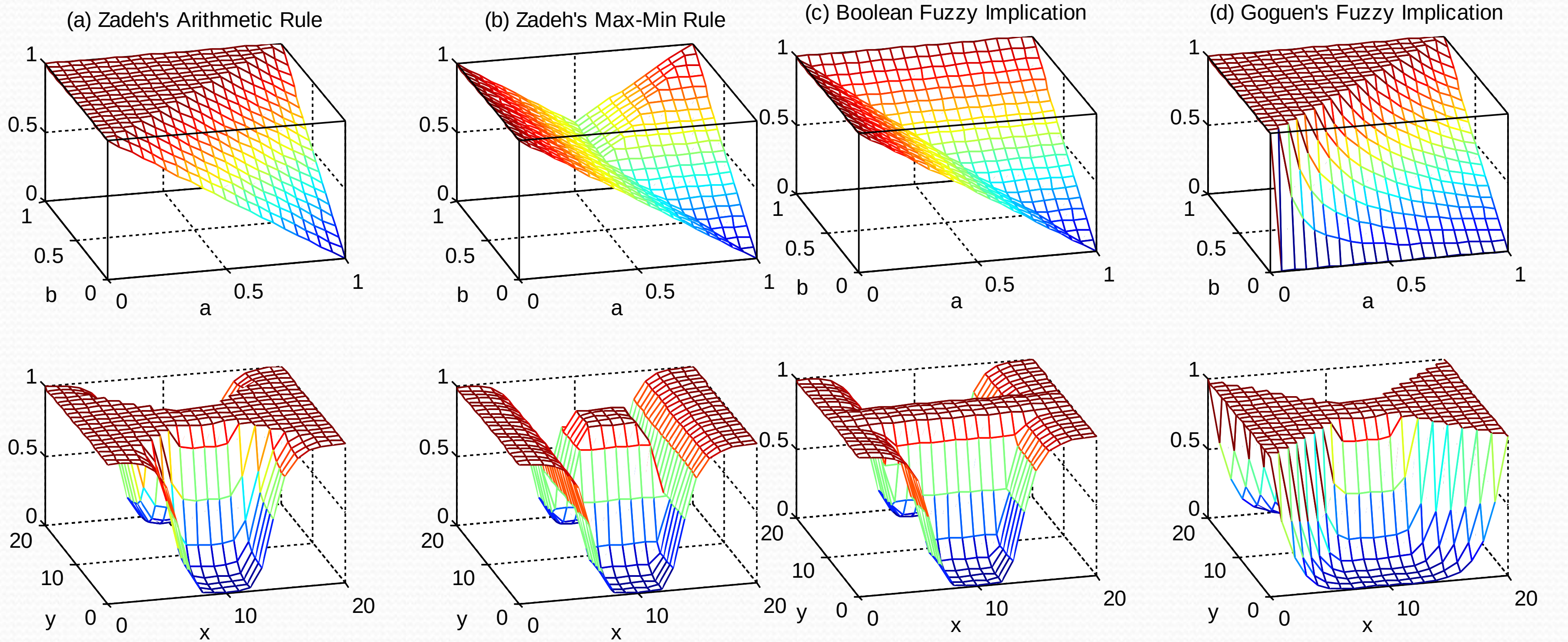
$$m_R(x,y) = max(1 - m_A(x), min(m_A(x), m_B(y)))$$

- Zadeh's arithmetic implication (based on $\neg A \vee B$)

$$m_R(x,y) = min(1 - m_A(x) + m_B(y), 1)$$

- Goguen's implication

$$m_R(x,y) = min(m_B(x) / m_A(y), 1)$$



(a) Zadeh's Arithmetic Rule
(b) Zadeh's Max-Min Rule
(c) Boolean Fuzzy Implication
(d) Goguen's Fuzzy Implication
a
b
x
y
0
0.5
1
10
20

Fuzzy inference systems

- Given a number of fuzzy rules:

    if temperature is low, then set heating high
    if air is dry, then set heating low

- If we do the observation

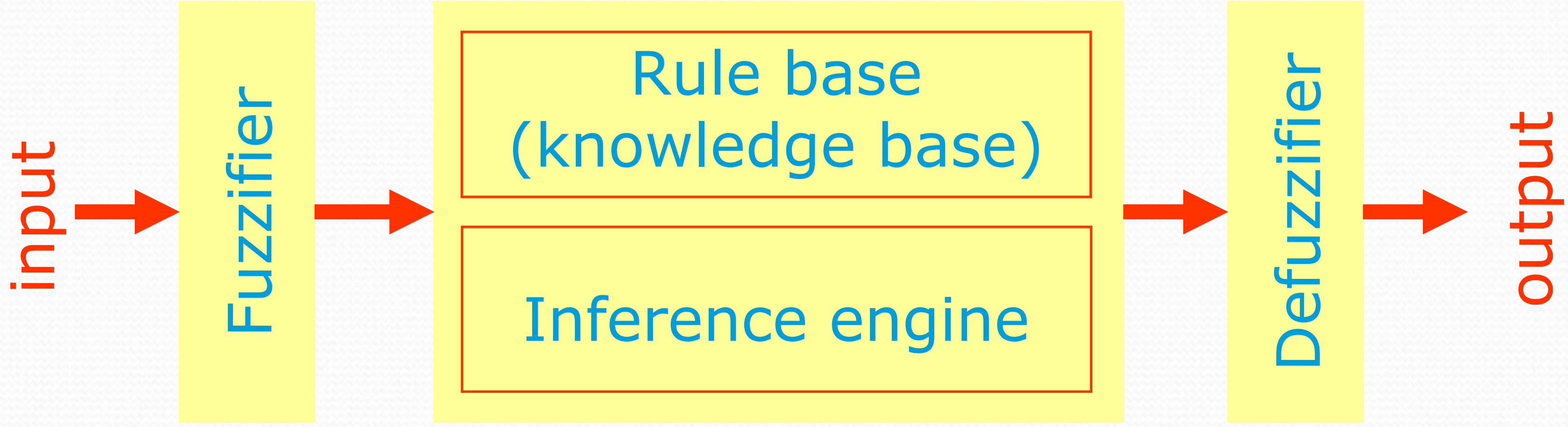
    temperature is 15c,
    air humidity 30%

  how do we set the heating?

- **Discussed here: Mamdani systems**

Building blocks

- Fuzzifier (in the simplest case, turn a measurement into a crisp set)
- Rule base
- Inference engine
- Defuzzifier

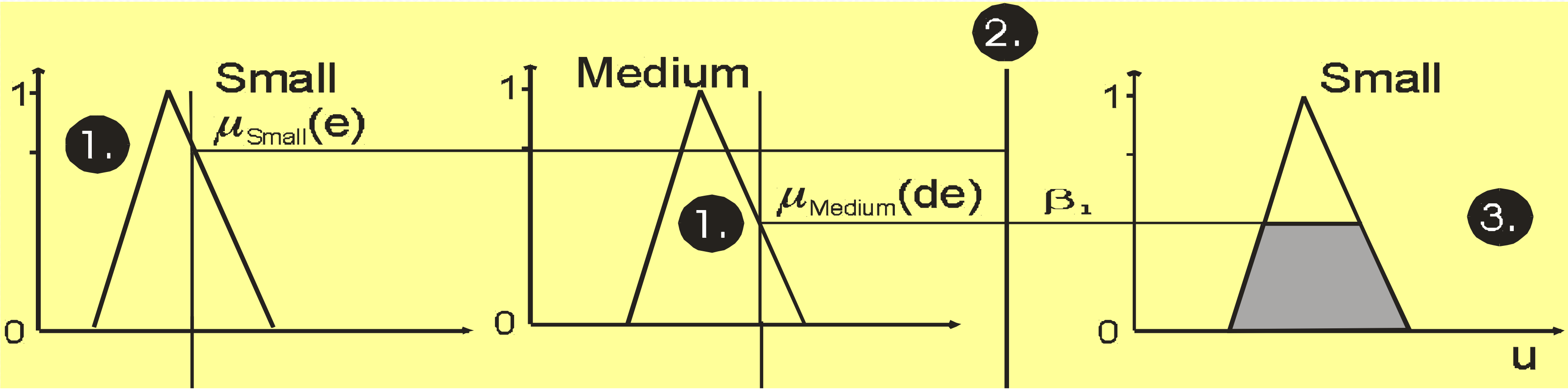


Mamdani Systems: Illustration on case 1

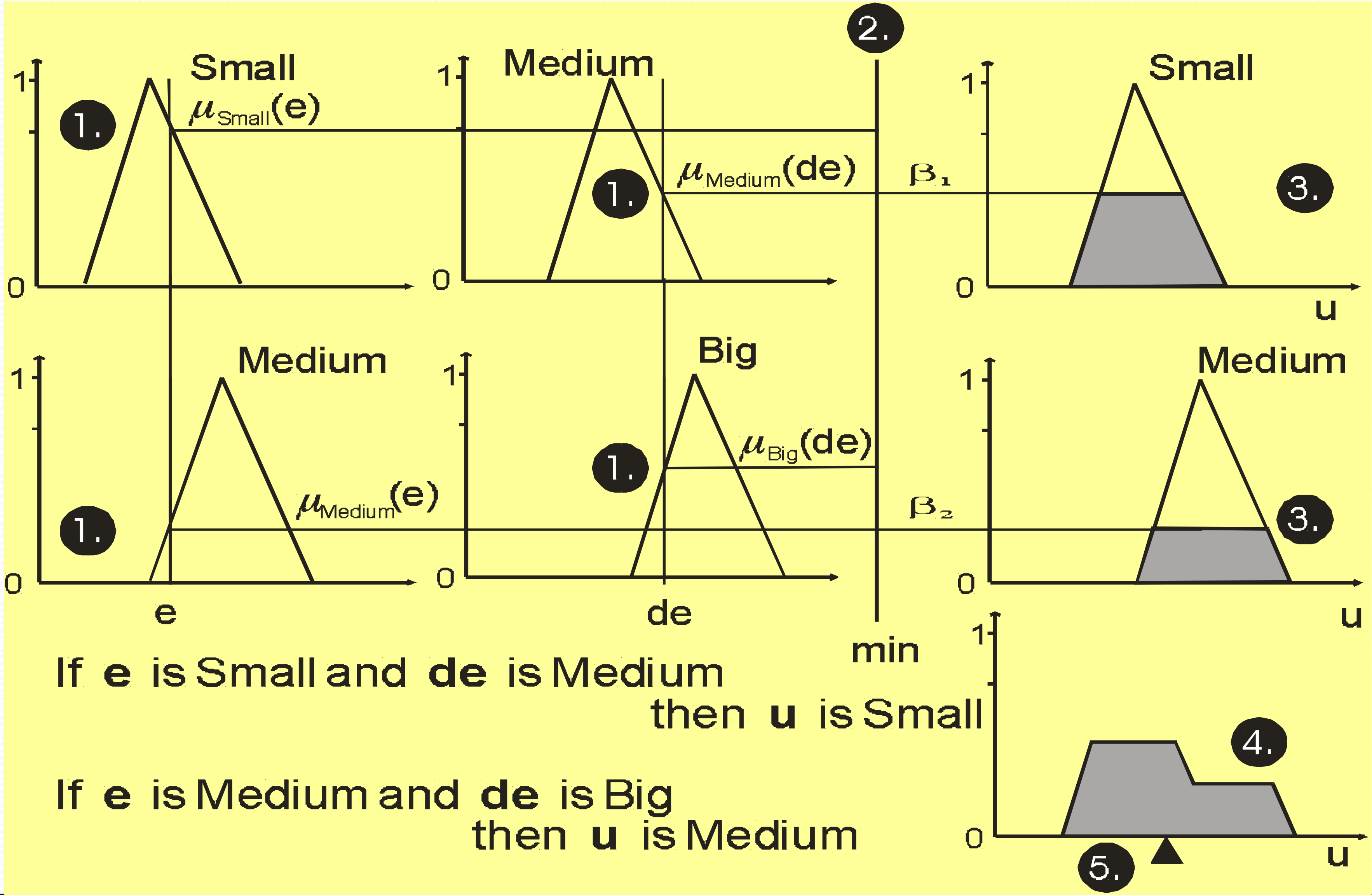
- When given are
  - a fuzzy rule A → B, where A and B are fuzzy sets defined by membership functions $\mu_A(x)$ and $\mu_B(y)$
  - a measurement *a* for A
- The membership function for A → B is defined by
  $\min(\mu_A(x), \mu_B(y))$
- For a measurement *a* the membership for *y* is $\min(\mu_A(a), \mu_B(y))$

Mamdani Systems: Illustration on case 2

- When rules contain multiple conditions, the min is taken over these conditions



Mamdani Systems: Illustration on case 3



Fuzzy observations

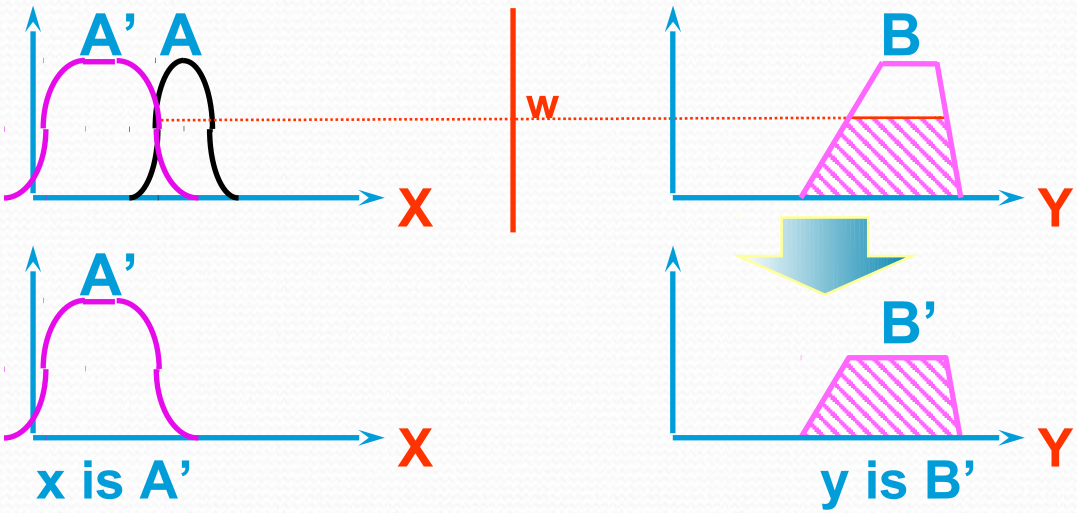
- Rule: if x is A then y is B
- Observation: x is A′ (fuzzy set)
- Conclusion: y is B′ (fuzzy set)

  defined as follows:

$$\mu_{B'}(y) = [\vee_x(\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y)$$

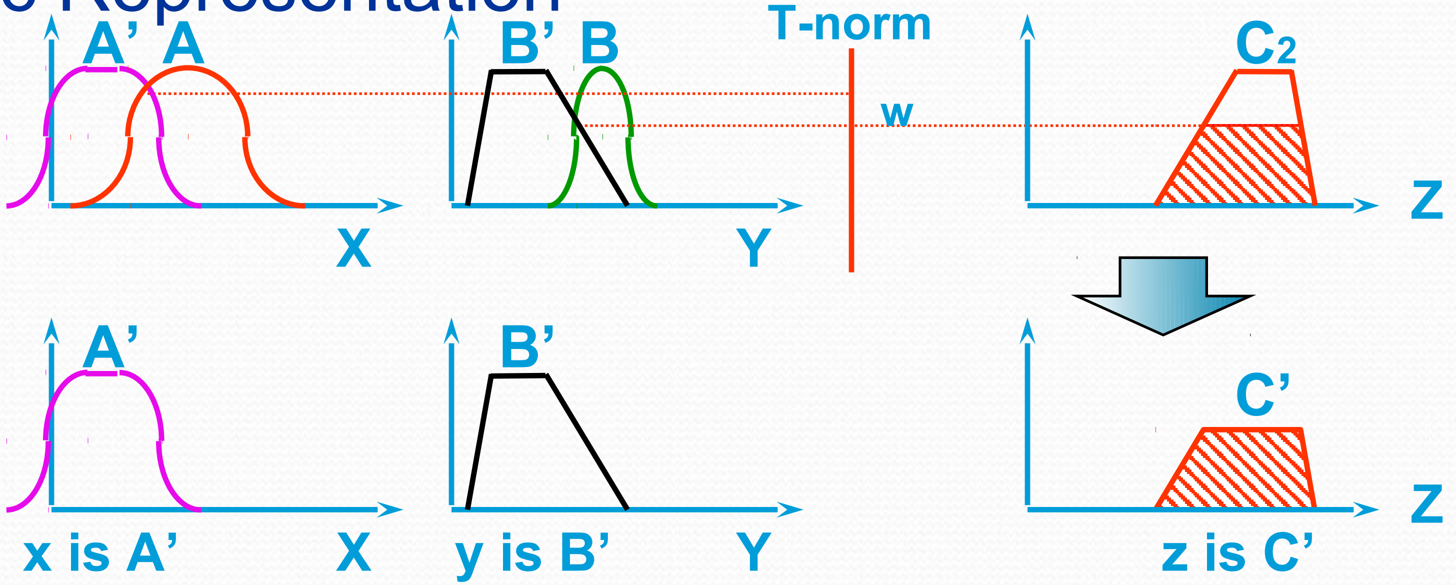
$$= w \wedge \mu_B(y)$$

Graphic Representation

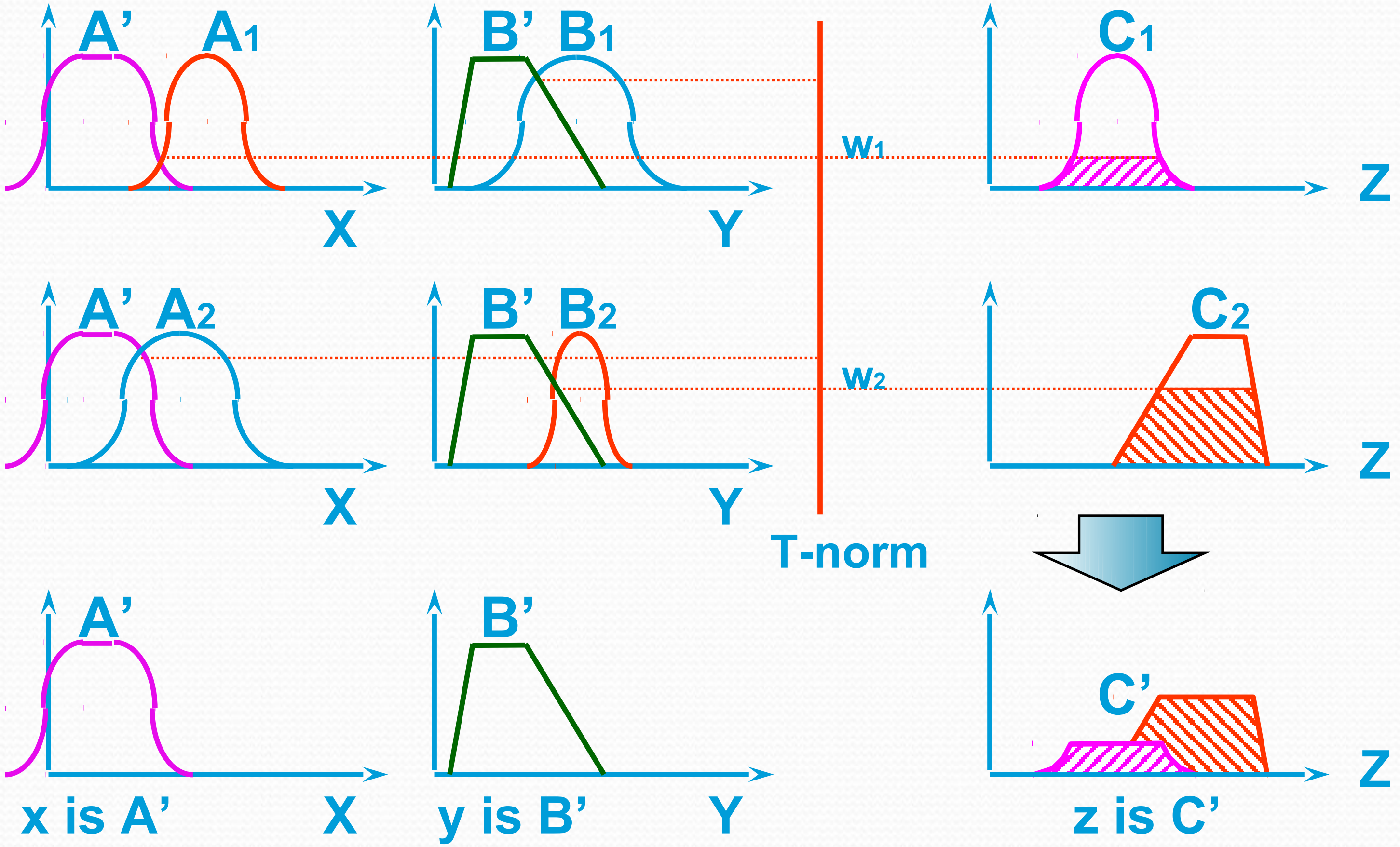


- Rule: if x is A and y is B then z is C
- Fact: x is A′ and y is B′
- Conclusion: z is C′

Graphic Representation



Multiple rules, multiple antecedents

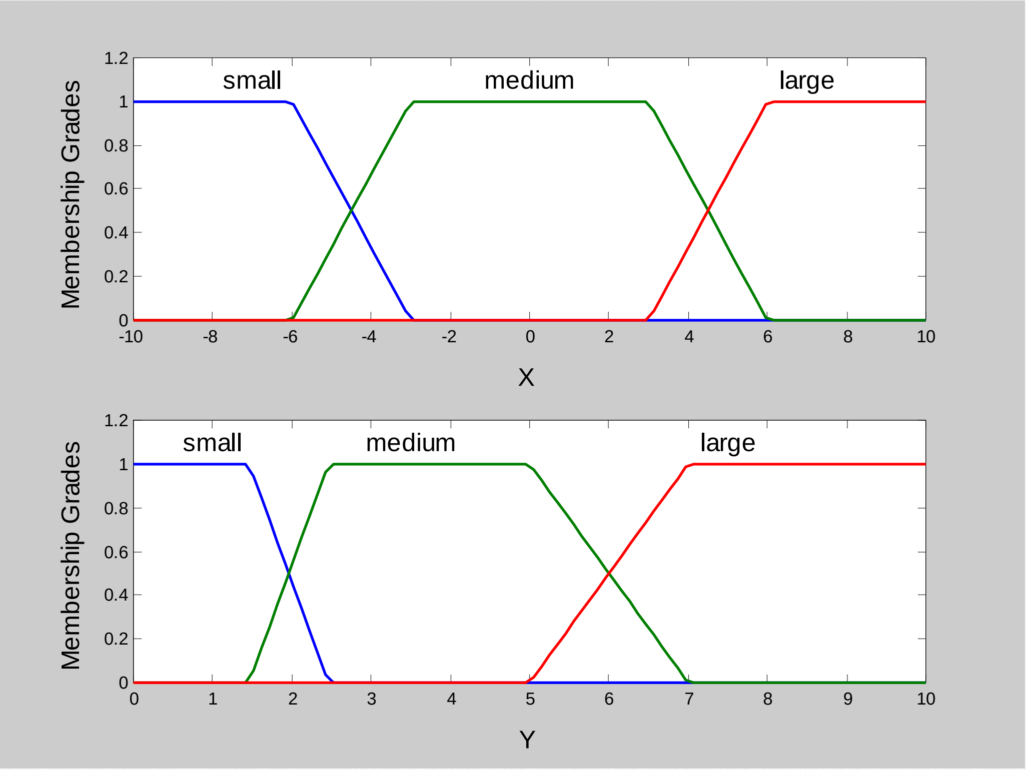


Defuzzification rules

- Centroid-of-area
$$z^* = \frac{\int_Z \mu_A(z)\, z\, dz}{\int_Z \mu_A(z)\, dz}$$
- Bisector of area
$$\int_{-\infty}^{z^*} \mu_A(z) dz = \int_{z^*}^{\infty} \mu_A(z) dz$$
- Mean of maximum
$$z^* = \frac{\int_{Z'} z\, dz}{\int_{Z'} dz}, \quad Z' = \{z \mid \mu_A(z) = \mu^*\}$$
- Smallest of maximum
$$\min_{z \in Z'} z$$
- Largest of maximum
$$\max_{z \in Z'} z$$

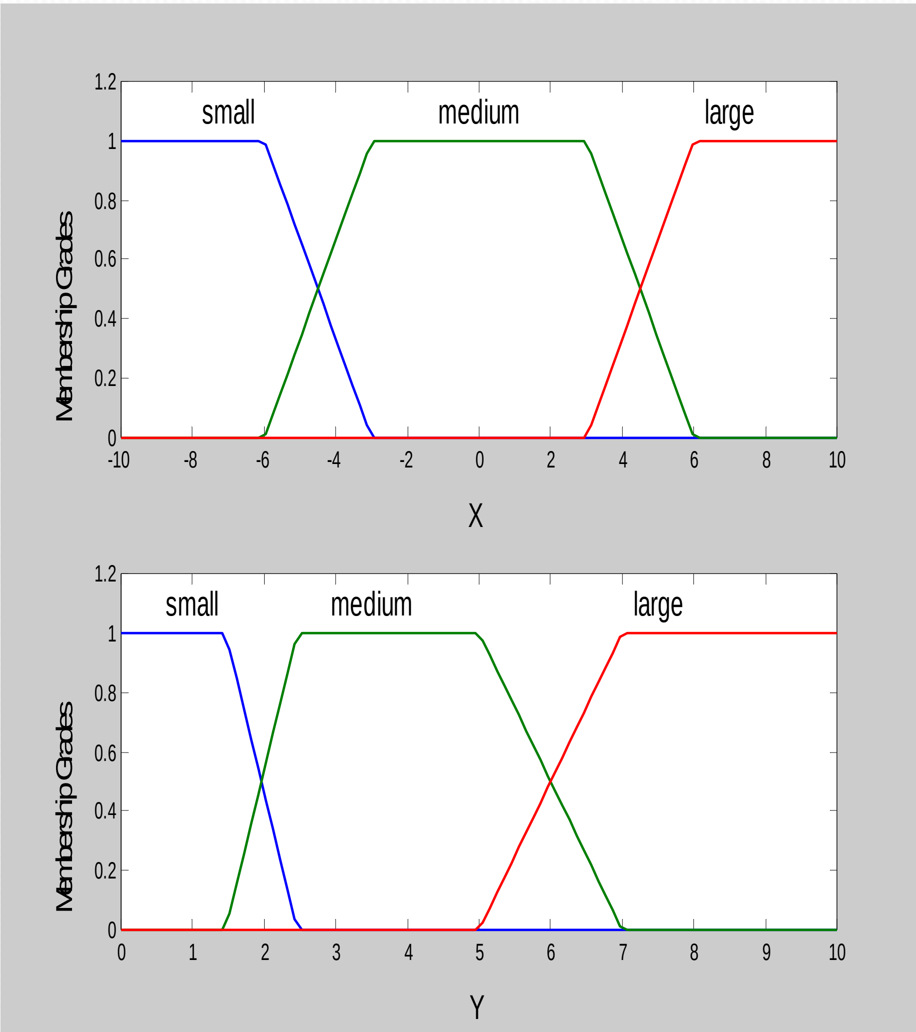
range of values where membership is maximal

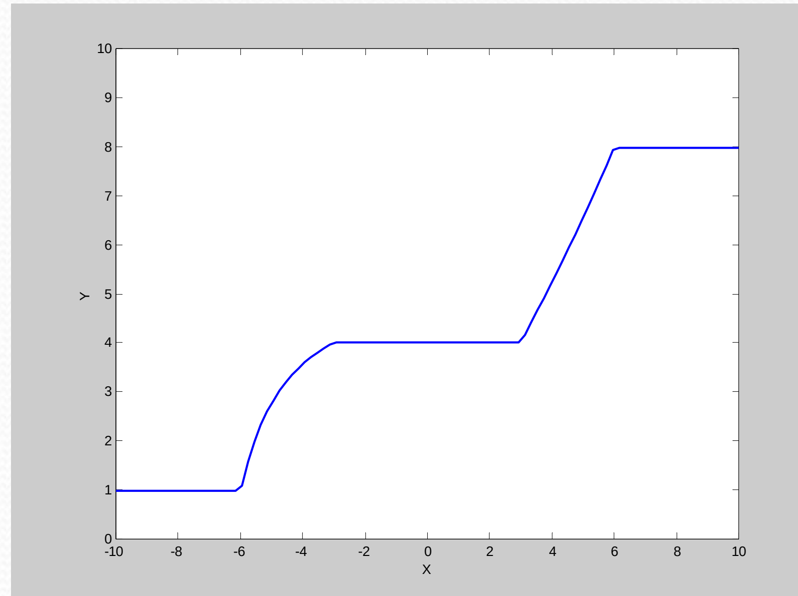
Mamdani - single input



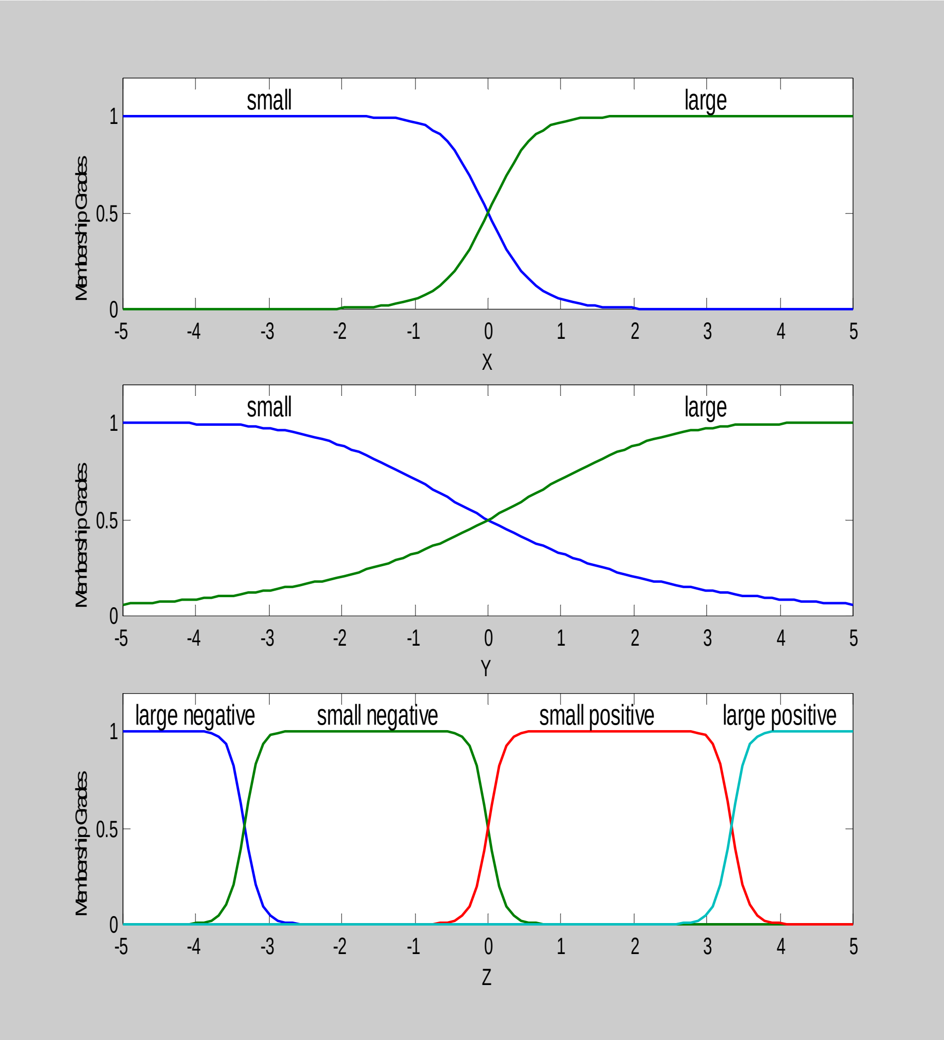
- X is Small → Y is Small
- X is Medium → Y is Medium
- X is Large → Y is Large

Mamdani - single input





Mamdani - double input



- X is Small and Y is Small → Z is negative Large
- X is Small and Y is Large → Z is negative Small
- X is Large and Y is Small → Z is positive Small
- if X is Large and Y is Large → Z is positive Large

Mamdani - double input